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A NOTE ON MATERIAL FRAME INDIFFERENCE

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Abstract—Three methods are presented by which the restrictions on the strain-energy density for an elastic material due to material frame indifference can be introduced. It is seen that one of these, which is extensively repeated in the secondary literature, contains a logical fallacy.

1. INTRODUCTION

In an elastic material which undergoes isothermal deformations the Helmholtz free energy per unit mass, or strain-energy density, is a function of the deformation gradient matrix only. It was shown by Green[1] in 1838 that it must depend on the latter through the six independent elements of the Cauchy strain matrix. Green's simple argument is outlined in Section 2. Since that time a number of other arguments which lead to the same result have been advanced. One of these is also given in Section 2.

In 1960 Truesdell[2] presented an argument, attributing it to Noll, which is superficially more attractive than the arguments previously given and has been repeated in the secondary literature by many authors, including one of the authors (R.S.R.) of this note.

Truesdell's argument has the additional attraction that it can be used to introduce into constitutive equations of the functional type the restriction implied by so-called material frame indifference on the manner in which the constitutive functional depends on the deformation gradient matrix history. Indeed it is in this context that the argument was advanced by Noll[3]. In Section 3 we draw attention to a logical fallacy in Truesdell's argument.

In the context of the present discussion, material frame indifference states that the strain-energy density W is independent of the choice of the rectangular coordinate system in which the deformed configuration is described. This is mathematically equivalent, again in the context of the present discussion, to the assumption that W is unaltered if an arbitrary rotation is superposed on the assumed deformation. It is in this sense that we interpret it in the present paper. We ignore the point, as irrelevant to the present discussion, that strictly material frame indifference allows the coordinate systems in which the deformed configuration is described to be either left-handed or right-handed.

2. TWO CORRECT ARGUMENTS

We consider an elastic material to undergo an isothermal deformation in which a particle P which initially has vector position $\mathbf{X}$ with respect to a fixed origin O moves to vector position $\mathbf{x}$ with respect to the same origin. Let X_A ($A = 1,2,3$) and x_i ($i = 1,2,3$) be the components of $\mathbf{X}$ and $\mathbf{x}$, respectively, in a fixed rectangular Cartesian coordinate system x with origin at O. The deformation gradient matrix $\mathbf{g}$ is defined by

$$\mathbf{g} = \|g_{iA}\| = \|\partial x_i/\partial X_A\|. \tag{1}$$

Let W be the Helmholtz free energy per unit mass. Since the material is elastic W is a function of $\mathbf{g}$ only

$$W = F(\mathbf{g}). \tag{2}$$

We now superpose on the assumed deformation an arbitrary rigid rotation, as a result of which the particle P moves to vector position $\bar{\mathbf{x}} = \mathbf{QX}$, where $\mathbf{Q}$ is a proper orthogonal matrix. Let $\bar{\mathbf{g}}$ be the deformation gradient matrix for the resultant deformation

$$\bar{\mathbf{g}} = \|\bar{g}_{iA}\| = \|\partial \bar{x}_i/\partial X_A\|. \tag{3}$$

The Helmholtz free energy per unit mass, $\bar{W}$, for the deformation $\mathbf{X} \to \bar{\mathbf{x}}$ is then given by

$$\bar{W} = F(\bar{\mathbf{g}}) = F(\mathbf{Qg}). \tag{4}$$

Since the Helmholtz free energy is unaltered by the superposed rotation $W = \bar{W}$ and hence, from eqns (2) and (4), the function F must satisfy the relation

$$F(\mathbf{g}) = F(\mathbf{Qg}) \tag{5}$$

for all proper orthogonal $\mathbf{Q}$. The restriction on the form of F implied by eqn (5) may be made explicit in the following manner.

Let $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ denote the first, second, and third columns of the matrix $\mathbf{g}$. Then, we may write

$$F(\mathbf{g}) = \hat{F}(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3). \tag{6}$$

With this definition of $\hat{F}$, eqn (5) becomes

$$\hat{F}(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) = \hat{F}(\mathbf{Qg}_1, \mathbf{Qg}_2, \mathbf{Qg}_3). \tag{7}$$

Equation (7) states that $\hat{F}$, and hence W, is a scalar invariant of the three vectors $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ under the proper orthogonal group. It must therefore be expressible as a function of the elements of a function basis for the three vectors $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ under the proper orthogonal group. Such a basis is provided by the six inner products

$$\mathbf{g}_A \cdot \mathbf{g}_B = C_{AB} \tag{8}$$

where

$$\mathbf{C} = \|C_{AB}\| \tag{9}$$

is the Cauchy strain matrix, and the scalar triple product

$$[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]. \tag{10}$$

We note that

$$[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]^2 = \det \mathbf{C}. \tag{11}$$

Since for a deformation which is possible in a real material

$$[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] > 0 \tag{12}$$

it follows that

$$[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] = (\det \mathbf{C})^{1/2}. \tag{13}$$

We conclude that $\hat{F}$ and hence $\mathbf{W}$ must be expressible as a function of the matrix $\mathbf{C}$

$$W = \hat{F}(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) = G(\mathbf{C}). \tag{14}$$

Since from eqns (8) and (9)

$$\mathbf{C} = \mathbf{g}^\dagger \mathbf{g} \tag{15}$$

where the dagger denotes the transpose, and

$$(\mathbf{Qg})^\dagger(\mathbf{Qg}) = \mathbf{g}^\dagger \mathbf{g} \tag{16}$$

for all proper orthogonal $\mathbf{Q}$, it follows from eqns (6) and (14) that if $F(\mathbf{g})$ has the form

$$F(\mathbf{g}) = G(\mathbf{C}) \tag{17}$$

it satisfies condition (5).

Equation (14) was first obtained by Green[1] by the following argument. A material element which initially has the form of a cuboid with its edges parallel to the axes of a rectangular Cartesian coordinate system x becomes, as a result of the deformation, an elementary parallelepiped. The ratios between the lengths of corresponding edges of the material element in the deformed and undeformed configurations and the angles between the edges in the deformed configuration are fully determined by C_{AB}. These six quantities fully describe the distortion which the material element undergoes in the deformation and are independent of the orientation of the material element in the deformed configuration. It follows that W is determined by $\mathbf{C}$.

3. AN INCORRECT ARGUMENT

Truesdell[2] has purported to reach the conclusions of the previous section by the following argument, which he attributes to Noll.

The Polar Decomposition Theorem enables us to decompose the deformation gradient matrix $\mathbf{g}$ in the form

$$\mathbf{g} = \mathbf{RU} \tag{18}$$

where $\mathbf{R}$ is a proper orthogonal matrix and $\mathbf{U}$ is a positive definite symmetric matrix. Then taking $\mathbf{Q} = \mathbf{R}^\dagger$ in eqn (5) Truesdell obtains, with eqn (18)

$$F(\mathbf{g}) = F(\mathbf{U}). \tag{19}$$

Since the polar decomposition of $\mathbf{Qg}$ is

$$\mathbf{Qg} = (\mathbf{QR})\mathbf{U} \tag{20}$$

it follows that

$$F(\mathbf{Qg}) = F(\mathbf{U}) \tag{21}$$

and hence, from eqns (19) and (21), so it is argued, the relation (cf. eqn (5))

$$F(\mathbf{g}) = F(\mathbf{Qg}) \tag{22}$$

is satisfied by any function of $\mathbf{U}$.

It is then maintained from eqn (18) that

$$\mathbf{U}^2 = \mathbf{C}. \tag{23}$$

Since **U** is positive definite, it is uniquely determined by **C** and hence W must be expressible in the form (cf. eqn (14))

$$W = G(\mathbf{C}). \tag{24}$$

The fallacy in Truesdell's argument may easily be seen from eqn (19). While it is true that any function $F(\mathbf{g})$ which satisfies eqn (5) must also satisfy eqn (19), it is not true that *any* $F(\mathbf{U})$ satisfies eqn (5). If it did it would also have to satisfy eqn (19). From eqn (18) $\mathbf{U} = \mathbf{R}^\dagger\mathbf{g}$ and it is evidently not true that any function of $\mathbf{R}^\dagger\mathbf{g}$ is the *same* function of **g**.

If eqn (5) is satisfied for all proper orthogonal **Q**, then

$$F(\mathbf{g}) = F(\mathbf{Q}\mathbf{g}) = F(\mathbf{P}\mathbf{Q}\mathbf{g}) \tag{25}$$

for all proper orthogonal **P**. Then, taking $\mathbf{Q} = \mathbf{R}^\dagger$ we obtain, with eqn (18)

$$F(\mathbf{U}) = F(\mathbf{P}\mathbf{U}) \tag{26}$$

for all proper orthogonal **P**. Plainly, not all functions of **U** satisfy this relation. To find those that do we can parallel the argument in Section 2 to obtain the result that $F(\mathbf{U})$ must be expressible in the form

$$F(\mathbf{U}) = G(\mathbf{U}^\dagger\mathbf{U}). \tag{27}$$

It is easily seen from eqn (18) that

$$\mathbf{U}^\dagger\mathbf{U} = \mathbf{g}^\dagger\mathbf{g} = \mathbf{C} \tag{28}$$

and we obtain immediately eqn (24).

We note further that in his argument Truesdell uses the matrix **U** in two slightly different senses. To see this we use indicial notation. Then eqn (18) may be written either as

$$g_{iA} = R_{iB}U_{BA} \tag{29}$$

or

$$g_{iA} = R_{ij}U_{jA} \tag{30}$$

where, of course

$$R_{ij} = R_{iB}\delta_{Bj}, \qquad U_{jA} = \delta_{jB}U_{BA}. \tag{31}$$

If eqn (19) is to have any meaning $\mathbf{U} = \|U_{iA}\|$ and eqn (19) becomes, in indicial notation

$$F(g_{iA}) = F(U_{iA}). \tag{32}$$

On the other hand if **U** is the positive definite symmetric matrix in the polar decomposition of **g**, then $\mathbf{U} = \|U_{AB}\|$. While $\|U_{iA}\|$ and $\|U_{AB}\|$ are numerically equal, they behave differently under the replacement $\mathbf{x} \to \mathbf{Q}\mathbf{x}$. In the notation of Schouten[4]

$$\|U_{iA}\| \stackrel{*}{=} \|U_{AB}\|. \tag{33}$$

While $\|U_{AB}\|$ is unchanged, $\|U_{iA}\|$ becomes $\|Q_{ij}U_{jA}\|$.

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